INFINITE-DIMENSIONAL GEOMETRY OF THE UNIVERSAL DEFORMATION OF THE COMPLEX DISK

Denis V. Yur'ev

July 11, 1992

Abstract. The universal deformation of the complex disk is studied from the viewpoint of infinite-dimensional geometry. The structure of a subsymmetric space on the universal deformation is described. The foliation of the universal deformation by subsymmetry mirrors is shown to determine a real polarization.

The subject of this paper may be of interest to specialists in algebraic geometry and representation theory as well as to researchers dealing with mathematical problems of modern quantum field theory.

The universal deformation of the complex disk is one of the crucial concepts used in the geometric statement of quantum conformal field theory [1] and quantum-field theory of strings [2] (see also [3]). The characteristic feature of the approach developed in the present paper is that the universal deformation of the complex disk is studied in terms of infinite-dimensional geometry. On this way the structure of the subsymmetric space [4–6] on the universal deformation is described. The foliation of the universal deformation defined by the mirrors of subsymmetries determines a real polarization. For a long time real polarizations on complex manifolds and their quantization have been attracting the attention of mathematicians dealing with algebraic geometry and representation theory and of specialists in mathematical physics [44–47]. The results of this paper confirm the importance of studying such polarizations and expose a connection between the traditions of classical synthetic geometry and recent trends in algebraic geometry, representation theory, and modern quantum field theory.

1. The infinite-dimensional geometry of the flag manifold of the Virasoro-Bott group (the base of the universal deformation of the complex disk).

1.1. The Virasoro algebra, the Virasoro-Bott group, and the Neretin semigroup. Let $\mathrm{Diff}(S^1)$ denote the group of diffeomorphisms of the unit circle S^1. The group manifold $\mathrm{Diff}(S^1)$ splits into two connected components, the subgroup $\mathrm{Diff}_+(S^1)$ and the coset $\mathrm{Diff}_-(S^1)$. The diffeomorphisms in $\mathrm{Diff}_+(S^1)$ preserve the orientation on the circle S^1 and those in $\mathrm{Diff}_-(S^1)$ reverse it.

The Lie algebra of $\mathrm{Diff}_+(S^1)$ can be identified with the linear space $\mathrm{Vect}(S^1)$ of smooth vector fields on the circle equipped with the commutator

$$[v(t)d/dt, u(t)d/dt] = (v(t)u'(t) - v'(t)u(t))d/dt. \tag{1}$$

In the basis

$$s_n = \sin(nt)d/dt, \quad c_n = \cos(nt)d/dt, \quad h = d/dt \tag{2}$$

the commutation relations have the form

$$
\begin{aligned}
[s_n, s_m] &= 0.5((m-n)s_{n+m} + \operatorname{sgn}(n-m)(n+m)s_{|n-m|}) \\
[c_n, c_m] &= 0.5((n-m)s_{n+m} + \operatorname{sgn}(n-m)(n+m)s_{|n-m|}) \\
[s_n, c_m] &= 0.5((m-n)c_{m+n} - (m+n)c_{|n-m|}) - n\delta_{nm}h \\
[h, s_n] &= nc_n \\
[h, c_n] &= ns_n
\end{aligned} \tag{3}
$$

The complexification of the Lie algebra $\mathrm{Vect}(S^1)$ will be denoted by $\mathrm{Vect}_{\mathbb{C}}(S^1)$. It is convenient to choose the following basis in $\mathrm{Vect}_{\mathbb{C}}(S^1)$:

$$e = ie^{ikt}d/dt \tag{4}$$

The commutation relations of the Lie algebra $\mathrm{Vect}_{\mathbb{C}}(S^1)$ have the following form

$$[e_j, e_k] = (j-k)e_{j+k} \tag{5}$$

in the basis e_k.

In 1968 I.M. Gelfand and D.B. Fuchs [7] discovered that $\mathrm{Vect}(S^1)$ possesses a non-trivial central extension. The corresponding 2-cocycle is

$$c(u,v) = \int v'(t)du'(t) \tag{6}$$

or, equivalently,

$$c(u,v) = \begin{vmatrix} v'(t_0) & u'(t_0) \\ v''(t_0) & u''(t_0) \end{vmatrix}. \tag{7}$$

This central extension was independently discovered by M. Virasoro [8] and named after him. Let us denote the Virasoro algebra by vir. Its complexification, which is also called the Virasoro algebra, will be denoted $\mathrm{vir}_{\mathbb{C}}$. As a vector space vir is generated by the vectors e_k and the central element c. The commutation relations have the form

$$[e_j, e_k] = (j-k)e_{j+k} + \delta(j+k)\frac{j^3-j}{12}c. \tag{8}$$

The infinite-dimensional group Vir corresponding to the Lie algebra vir is a central extension of the group $\mathrm{Diff}(S^1)$. The corresponding 2-cocycle was calculated by R. Bott [9]. This cocycle can be written as

$$c(g_1, g_2) = \int \log(g_1' \circ g_2)\log(g_2'). \tag{9}$$

The group Vir is called the Virasoro–Bott group.

There are no groups corresponding to the Lie algebras $\mathrm{Vect}_{\mathbb{C}}(S^1)$ or $\mathrm{vir}_{\mathbb{C}}$, but one can consider the following construction due to Yu. A. Neretin, M. L. Kontzevich, and G. Segal.

Let us denote by $\mathrm{L}\,Diff_+^{\mathbb{C}}(S^1)$ the set of all analytic mappings $g : S^1 \mapsto \mathbb{C}\backslash\{0\}$ such that $g(S^1)$ is a Jordan curve surrounding zero, the orientations of S^1 and $g(S^1)$ are the same, and $g'(e^{i\theta})$ is everywhere different from zero. $\mathrm{L}\,Diff_+^{\mathbb{C}}(S^1)$ is a local group [10].

Let $\mathrm{LNer} \subset \mathrm{L}\,Diff_+^{\mathbb{C}}(S^1)$ be the local subsemigroup of mappings g such that $|g(e^{i\theta})| < 1$. As was shown by Yu. A. Neretin [10], the structure of a local semigroup on LNer extends to the structure of a global semigroup Ner.

There exist at least two constructions of the semigroup Ner.

The first construction (Yu. A. Neretin [10]). An element of Ner is a formal product

$$p \cdot A(t) \cdot q \tag{10}$$

where $p, q \in \mathrm{D}\,iff_+(S^1)$, $p(1) = 1$, $t > 0$, $A(t) : C \mapsto C$, $A(t)z = e^{-t}z$.

To define the multiplication in Ner one must describe the rule to transform the formal product $A(s)\cdot p\cdot A(t)$ to the form (10).

A. Let t be so small that the diffeomorphism p extends holomorphically to the annulus $e^{-t}\leqslant |z|\leqslant 1$. Then the product $g=A(s)pA(t)$ is well-defined. Let K be the domain bounded by S^1 and $g(S^1)$. Let Q be the canonical conformal mapping of K onto the annulus $e^{-t'}\leqslant |z|\leqslant 1$, normalized by the condition $Q(1)=1$. Then $g=p'\cdot A(t)\cdot q'$, where $p'=Q^{-1}|_{S^1}$ and q' is determined by the identity

$$A(s)\cdot p\cdot A(t)=p'\cdot A(t')\cdot q'. \tag{11}$$

B. For an arbitrary t there exists a suitable n such that the product

$$A(s)\cdot p\cdot A(t)=(\dots(A(s)\cdot p\cdot A(t/n))A(t/n)\dots)A(t/n) \tag{12}$$

can be calculated. It can be shown that the product does not depend on the choice of the representation (12) and is associative [10].

The second construction (M. L. Kontsevich [11] and G. Segal [12]). An element g of the semigroup Ner is a triple (K,p,q), where K is a Riemann surface with boundary ∂K such that K is biholomorphically equivalent to an annulus and $p,q: S^1\mapsto\partial K$ are fixed parametrizations of the components of ∂K. Two elements $g_i=(K_i,p_i,q_i)$, $i=1,2$, are equivalent if there exists a conformal mapping $R:K\mapsto K$ such that $p_2=Rp_1$, and $q_2=Rq_1$. The product of two elements g_1 and g_2 is the element $g_3=(K_3,p_3,q_3)$, where

$$K_3=K_1\bigsqcup_{q_1(e^{it})=p_2(e^{it})}K_2,$$

$p_3=p_1$, and $q_3=q_2$.

This construction admits a slight modification [13]. Let us consider the semigroup $\overline{\mathrm{Ner}}$ whose elements g are pairs (p_g^+,p_g^-), where $p_g^\pm: D_\pm\mapsto\mathbb{C}$, $D_\pm=\{z:|z|^{\pm1}\leqslant 1\}$, such that $p_g^+(D_+^0)\cap p_g^-(D_-^0)=\varnothing$. Two elements g_1 and g_2 are equivalent if there exists a biholomorphic mapping $R:\overline{\mathbb{C}}\mapsto\overline{\mathbb{C}}$ such that $p_{g_2}^+=Rp_{g_1}^+$ and $p_{g_2}^-=Rp_{g_1}^-$. The product is defined by analogy with the previous construction.

The Neretin semigroup $\bar{\mathrm{N}}\mathrm{er}$ possesses a central extension. The corresponding cocycle was calculated by Yu. A. Neretin [13]:

$$\begin{aligned}c(g_1,g_2)=&\oint\log(p_2^+)'(z)d\log\frac{(p_3^+)'(g_1(z))}{(p_1^+)'(g_1(z))}-\oint\frac{\log(p_2^+)(z)}{z}d\log\frac{p_3^+(g_1(z))}{p_1^+(g_1(z))}\\&+\oint\log(p_2^-)(g_2(z))d\log\frac{p_3^-(z)}{p_1^-(z)}-\oint\log\frac{p_2^-(g_2(z))}{g_2(z)}d\log\frac{p_3^-(z)}{p_1^-(z)}.\end{aligned} \tag{13}$$

1.2. The flag manifold of the Virasoro-Bott group. The flag manifold M of the Virasoro-Bott group is a homogeneous space with transformation group $\mathrm{Diff}_+(S^1)$ and isotropy group S^1. There exist several different realizations of this manifold [14–18].

Algebraic realization. The space M can be realized as a conjugacy class in the group $\mathrm{Diff}_+(S^1)$ or in the Virasoro-Bott group Vir [15]. It should be mentioned that M can also be realized as the quotient $\mathrm{Ner}/\mathrm{Ner}^\circ$ of the Neretin semigroup by the subsemigroup Ner° of elements admiting holomorphic extension to D_-.

Probabilistic realization. Let P be the space of real probability measures $\mu=u(t)\,dt$ with smooth positive density $u(t)$ on S^1. The group $\mathrm{Diff}_+(S^1)$ naturally acts on P by the formula

$$g\colon u(t)\,dt\mapsto u(g^{-1}(t))\,dg^{-1}(t). \tag{14}$$

The action is transitive and the stabilizer of the point $(2\pi)^{-1}\,dt$ is isomorphic to S^1. Hence, P can be identified with M.

Orbital realization. The space M can be considered as an orbit of the coadjoint representation of $\mathrm{Diff}_+(S^1)$ or Vir [17, 18]. Namely, the elements of the dual space vir* of the Virasoro algebra vir can be identified with the pairs $(p(t)\,dt^2,b)$; the coadjoint action of Vir has the form

$$K(g)(p,b)=(gp-bS(g),b), \tag{15}$$

where

$$S(g) = \frac{g'''}{g'} - \frac{3}{2}\Big(\frac{g''}{g'}\Big)^2$$

is the Schwarzian (the Schwarz derivative). The orbit of the point $(a \cdot dt^2, b)$ coincides with M provided that $a/b \neq -n^2/2$, $n = 1, 2, 3, \ldots$. Therefore, a family $\omega_{a,b}$ of symplectic structures is defined on M.

Analytic realization. Let us consider the space S of univalent functions on the unit disk D_+ [19–21]. The Taylor coefficients $c_1, c_2, c_3, c_4, \ldots$ in the expansion

$$f(z) = z + c_1 z^2 + c_2 z^3 + c_3 z^4 + \cdots + c_n z^{n+1} \ldots \tag{16}$$

form a coordinate system on S. It was shown in [14] that S can be naturally identified with M. In the coordinate system $\{c_k\}$ the action of the Lie algebra $\mathrm{Vect}_{\mathbb{C}}(S^1)$ on M has the following form:

$$\mathfrak{L}_v(f(z)) = if^2(z) \oint \Big(\frac{wf'(w)}{f(w)}\Big)^2 \frac{v(w)}{f(w) - f(z)} \frac{dw}{w} \tag{17A}$$

or

$$\begin{aligned}
L_p &= \frac{a}{ac_p} + \sum_{k\geqslant 1} (k+1)c_k \frac{a}{ac_{k+p}}, \quad p > 0 \\
L_0 &= \sum_{k\geqslant 1} kc_k \frac{a}{ac_k} \\
L_{-1} &= \sum_{k\geqslant 1} ((k+2)c_{k+1}) \frac{a}{ac_k} \\
L_{-2} &= \sum_{k\geqslant 1} ((k+3)c_{k+2}) - (4c^2 - c_1^2)c_k - B_k) \frac{a}{ac_k} \\
L_{-p} &= \frac{(-1)^p}{(p-2)!} ad^{p-2}(L_{-1})L_{-2}
\end{aligned} \tag{17B}$$

where B_k are the Laurent coefficients of the function $1/(wf(w))$. The symplectic structure $\omega_{a,b}$ together with the complex structure on M determines a Kähler metric $w_{a,b}$. More detailed information can be found in [3, 15, 16, 22, 23].

It should be mentioned that the space M can be realized as a space of complex structures on loop manifolds [24].

1.3. Non-Euclidean geometry of mirrors. Points and Lagrangian submanifolds are the basic elements of symplectic geometry [25-28]. However the space of all Lagrangian submanifolds is infinite-dimensional and hard to visualize, which is not convenient. The Kähler geometry on the flag manifold M of the Virasoro-Bott group permits us to select a handy subset in the set of all Lagrangian submanifolds [29].

By a Kähler subsymmetry [29, 4–6] we mean an involutory antiautomorphism of M. The set of fixed points of a subsymmetry (a *mirror*) is a completely geodesic Lagrangian submanifold. Points and mirrors are the basic elements of the geometry to be described. The set of all mirrors forms a symmetric space, the non-Euclidean Lagrangian (Lagrange Grassmannian) $\Lambda(M)$, independent of a choice of the non-Einsteinian Kähler $\mathrm{Diff}_+(S^1)$-invariant metric $w_{a,b}$ $(b/a \neq -13)$ on the space M [24, 3, 29].

Consider the probabilistic realization of M and the set $A(M)$ of measures of the form $\delta_a(t)dt$. The set $A(M)$ is called the absolute [3, 29]. The absolute $A(M)$ is isomorphic to S^1. Let us introduce the parallelism relation on the non-Euclidean Lagrangian. Two mirrors are said to be parallel if they pass through the same point of the absolute. The following analog of the Lobachevskii axiom holds: for any point of M and any point of $A(M)$ there exists exactly one mirror passing through these points.

Let us also introduce the non-Euclidean Lagrangian $\Lambda_+(M)$ [29]. The elements of $\Lambda_+(M)$ are the oriented mirrors V_a. An oriented mirror is a pair (V, a), where V is a mirror and $a \in V$ a point of the absolute.

It should also be mentioned that the Maslov index $m(U_a, V_b, W_c)$ [30] of three oriented mirrors U_a, V_b, and W_c is equal to 1 if the orientations of the triple (a, b, c) and of the circle $S^1 \simeq A(M)$ are the same and to -1 if these orientations are opposite.

2. The infinite-dimensional geometry of the skeleton of the flag manifold of the Virasoro-Bott group

2.1. An equivariant mapping of the flag manifold of the Virasoro-Bott group into the infinite-dimensional classical domain of third type. Let H_0 be the space of smooth real-valued 1-forms $u(\exp(it))\,dt$ on the circle such that

$$\int u(\exp(it))\,dt = 0.$$

Let $H_0^{\mathbb{C}}$ be its complexification, and let $(H_0^{\mathbb{C}})_+$ and $(H_0^{\mathbb{C}})_-$ be the transversal spaces consisting of 1-forms possessing holomorphic extensions to the disks D_+ and D_-, respectively. Let $\mathcal{O}(S^1)$, $\mathcal{O}(D_+)$, and $\mathcal{O}(D_-)$ denote the spaces of holomorphic functions on S^1, D_+, and D_-, respectively. Then the space $H_0^{\mathbb{C}}$ is isomorphic to the quotient of $\mathcal{O}(S^1)$ by constants:

$$f(z) \in \mathcal{O}(S^1) \mapsto df(z) \in H_0^{\mathbb{C}}.$$

Under this isomorphism the spaces $(H_0^{\mathbb{C}})_+$ and $(H_0^{\mathbb{C}})_-$ are identified with the quotients of $\mathcal{O}(D_+)$ and $\mathcal{O}(D_-)$, respectively, by constants. Consider the completion $H^{\mathbb{C}}$ of the space $H_0^{\mathbb{C}}$ with respect to the norm

$$\|u\| = \sum_n |u_n|^2/n, \quad u(z) = \sum_n u_n z^n,$$

and let $H_+^{\mathbb{C}}$ and $H_-^{\mathbb{C}}$ be the corresponding completions of $(H_0^{\mathbb{C}})_+$ and $(H_0^{\mathbb{C}})_-$. The space $H^{\mathbb{C}}$ is equipped with the symplectic and pseudo-Hermitian structures defined by

$$\begin{aligned} (f(z), g(z)) &= \oint f(z)\,dg(z), \\ \langle f(z), g(z)\rangle &= \oint f(z)\,\overline{dg(z)}, \quad f, g \in \mathcal{O}(S^1). \end{aligned} \tag{18}$$

Let us denote the invariance groups of these structures by $\mathrm{Sp}(H^{\mathbb{C}}, \mathbb{C})$ and $U(H_+^{\mathbb{C}}, H_-^{\mathbb{C}})$, respectively; also, let $\mathrm{Sp}(H, \mathbb{R}) = \mathrm{Sp}(H^{\mathbb{C}}, \mathbb{C}) \cap U(H_+, H_-)$.

Consider the Grassmannian $\mathrm{Gr}(H^{\mathbb{C}})$, that is, the set of all complex Lagrangian subspaces in $H^{\mathbb{C}}$ [48, 42]. The space $\mathrm{Gr}(H^{\mathbb{C}})$ is an infinite-dimensional homogeneous space with transformation group $\mathrm{Sp}(H^{\mathbb{C}}, \mathbb{C})$. Consider the action of the subgroup $\mathrm{Sp}(H, \mathbb{R})$ on $\mathrm{Gr}(H^{\mathbb{C}})$. The orbit of the point $H_-^{\mathbb{C}}$ is an open (in a suitable topology) subspace $\mathcal{R}$ in $\mathrm{Gr}(H^{\mathbb{C}})$ isomorphic to $\mathrm{Sp}(H, \mathbb{R})/U$, where U is the group of operators on $H^{\mathbb{C}} = H_+^{\mathbb{C}} \oplus H_-^{\mathbb{C}}$ of the form $A \oplus \widetilde{A}$, $A \in U(H_+^{\mathbb{C}})$, $\widetilde{A} \in U(H_-^{\mathbb{C}})$; here the mapping $A \mapsto \widetilde{A}$ from $U(H_+^{\mathbb{C}})$ to $U(H_-^{\mathbb{C}})$ is induced by the inversion $z \mapsto 1/z$.

The manifold $\mathcal{R}$ is an infinite-dimensional classical homogeneous domain of third type [31]. The manifold $\mathcal{R}$ is mapped in the linear space $\mathrm{Hom}(H_-^{\mathbb{C}}, H_+^{\mathbb{C}})$ in such a way that the elements of $\mathcal{R}$ are represented by symmetric matrices Z satisfying $E - Z\bar{Z} > 0$.

The mapping of M into $\mathcal{R}$ is described in [16, 22, 23, 3]. Namely, the representation of $\mathrm{Diff}_+(S^1)$ in H defines a monomorphism $\mathrm{Diff}_+(S^1) \mapsto \mathrm{Sp}(H, \mathbb{R})$. Hence, $\mathrm{Diff}_+(S^1)$ acts on $\mathcal{R}$. The orbit of the initial point under this action coincides with $\mathrm{Diff}_+(S^1)/\,\mathrm{PSl}(2, \mathbb{R})$. Therefore, we have a mapping $M \mapsto \mathcal{R}$. The explicit form of this mapping can be found in [16, 23] (see [3]). The matrix Z_f corresponding to a univalent function $f \in S$ is called the Grunskii matrix, and the mapping $S \simeq M \mapsto \mathcal{R} \hookrightarrow \mathrm{Gr}(H^{\mathbb{C}})$ is the Krichever mapping [41, 42].

It is well known [32] that the skeleton of a finite-dimensional classical domain of third type consists of all symmetric unitary matrices. Thus we can regard the set of all symmetric unitary operators from H to H as the skeleton of $\mathcal{R}$. By the skeleton of the space S of univalent functions is defined as the set of functions whose Grunskii matrices are unitary. Accordingly to Milin's theorem [33, 34], the skeleton of S consists of all univalent functions f such that $\mathrm{mes}(\mathbb{C} \setminus f(D_+^0)) = 0$. Let us investigate the structure of S more systematically.

2.2. The geometry of the skeleton of the space S. Consider the $\mathbb{R}$-analytic space E whose elements are cuts of the complex plane $\mathbb{C}$ with one end at infinity such that the conformal radius of $\mathbb{C} \setminus K$ with respect to zero is equal to one. Consider the mapping $E \to \Lambda_+(M)$ defined as

$$f(z) \mapsto (s, a), \tag{19}$$

where

$$\begin{gathered} f(D_+^0) \sqcup K = \bar{C}, \quad f(0) = 0, \quad f'(0) = 1, \\ f(a) = \infty, \quad f(s(z)) = f(z). \end{gathered} \tag{19$'$}$$

Theorem 1A. $E \simeq \Lambda_+(M)$.

Proof. Note that $\Lambda_+(M) = \{(s,a) \colon a \in \mathrm{Diff}_-(S^1),\ s^2 = id,\ a \in S^1,\ s(a) = a\}$. It is clear that the mapping (19) is an embedding. Let us prove that it is a surjection. To this end consider an arbitrary element (s,a) of $\Lambda_+(M)$ and construct the manifold $D_+^s = D_+/(z = s(z))$. Then D_+^s is topologically equivalent to the Riemann sphere, and therefore D_+^s and $\overline{C}$ are equivalent as complex manifolds. Hence, there exists unique mapping $f \colon D_+^s \mapsto \mathbb{C}$ such that $f(0) = 0$, $f'(0) = 1$, and $f(a) = \infty$. The composition of f with the natural projection $D_+ \to D_+^s$ is a function representing the element of E corresponding to the pair (s,a).

Let us now embed E in the skeleton of S. Note that the group $\mathrm{Diff}(S^1)$ naturally acts on the skeleton. On the other hand, Theorem 1A defines the structure of a $\mathrm{Diff}_+(S^1)$-homogeneous space on E. The question is whether the embedding of E in the skeleton of S is $\mathrm{Diff}_+(S^1)$-equivariant.

Theorem 1B. *The embedding of $E \simeq \Lambda_+(M)$ in the skeleton of S is $\mathrm{Diff}_+(S^1)$-equivariant.*

Proof. Note that the action of $\mathrm{Diff}_+(S^1)$ on the skeleton of S can be reduced to the subspace E. Namely, formulas (17A) that define the infinitesimal action correspond to analytic variations of cuts in E. One obtains the action of $\mathrm{Diff}_+(S^1)$ on E by exponentiating these deformations.

It is necessary to verify that this action is the same as defined in the theorem.

Note that the group $\mathrm{Diff}(S^1)$ naturally acts on $\mathrm{Gr}(H^{\mathbb{C}})$ preserving R and its skeleton. On the other hand, $\mathrm{Diff}(S^1)$ preserves $S \simeq \mathrm{Diff}_+(S^1)/S^1$ and therefore preserves its skeleton. This statement is true for each Kähler subsymmetry of S (which is an involutory element of $\mathrm{Diff}_-(S^1)$). Consider an arbitrary subsymmetry $s \in \Lambda(M)$. Its mirror V_s extends analytically to be an element of $\mathrm{Gr}(H^{\mathbb{C}})$. The set $V_s \cap E$ consists of exactly two points that correspond to different elements (s,a) and (s,b) of $\Lambda_+(M)$ under the isomorphism (19). Thus, the cited actions of $\mathrm{Diff}_+(S^1)$ on E are the same.

Note that $\Lambda_+(M)$ is a symmetric space,

$$(s_1,a_1)(s_2,a_2) = (s_1 s_2 s_1, s_1(a_2)),$$

with trasvection group $\mathrm{Diff}_+(S^1)$ and isotropy group $G_0 = \{g \in \mathrm{Diff}(S^1) \colon g(1) = 1,\ \overline{g(z)} = g(\bar{z})\}$. The tangent space V to $\Lambda_+(M)$ at the point $(s_-,1)$, $s_-(z) = \bar{z}$, can be identified with the space of odd vector fields on S^1. The directions in V invariant with respect to G_0 and determined by the generalized vectors $\delta_1(t)d/dt$ and $\delta_{-1}(t)d/dt$ give rise to nonholonomic generalized invariant direction fields ξ_+ and ξ_- on $\Lambda_+(M)$. Let $\mathcal{O}(E)$ and $\mathcal{O}(\Lambda_+(M))$ be the structure rings of E and $\Lambda_+(M)$, respectively; we have $\mathcal{O}(\Lambda_+(M)/\xi_-) = \{f \in \mathcal{O}(\Lambda_+(M)) \colon \xi_- f = 0\}$.

Theorem 1C. $(E, \mathcal{O}(E)) \simeq (\Lambda_+(M), \mathcal{O}(\Lambda_+(M)/\xi_-))$.

Proof. As was mentioned, the tangent space to $\Lambda_+(M)$ at the point $(s_-,1)$, $s_-(z) = \bar{z}$, can be identified with the space of odd vector fields on S^1. Under the isomorphism (19) the point $(s_-,1) \in \Lambda_+(M)$ corresponds to the Koebe function [19, 2]

$$k(z) = z/(1-z)^2.$$

The Killing fields on E defined by (17A) and vanishing at the point $k(z)$ are just the odd vector fields and ξ_-. Since E and $\Lambda_+(M)$ are homogeneous spaces, it follows that $\mathcal{O}(E) = \mathcal{O}(\Lambda_+(M)/\xi_-)$.

Let us now consider the mapping $M \mapsto \Gamma_{cl}(\Lambda_+(M))$, where $\Gamma_{cl}(\Lambda_+(M))$ is the space of all closed geodesics on $\Lambda_+(M)$, that assigns to each point $x \in M$ the set of all oriented mirrors passing through x. This mapping is clearly an isomorphism. Namely, consider a closed geodesic on $\Lambda_+(M)$. This geodesic is a symmetric space with transvection group isomorphic to some subgroup $S^1 \subset \mathrm{Diff}_+(S^1)$. This subgroup is the stabilizer of a suitable point of the flag manifold M.

Under the identification of M and $\Gamma_{cl}(\Lambda_+(M))$ the symplectic structure on M has the form

$$\omega_x(X,Y) = \int_{\gamma_x} (AX,Y)\, d\tau, \tag{20}$$

where τ is the natural parameter on γ_x and X, Y are Jacobi fields orthogonal to the field $\dot{\tau}$ in the unique (up to a real factor) invariant degenerate pseudo-Riemannian metric on $\Lambda_+(M)$; $A = a\nabla + b\nabla^3$, where ∇ is the covariant derivative along γ_x (cf. [35]).

Consider the class $\mathcal{O}^0(S)$ of holomorphic functionals on S that admit analytic extension to E.

Theorem 2. $\Re(\mathcal{O}^0(S)) \simeq \mathcal{O}(\Lambda_+(M)/\xi_-)$. *The isomorphism is given by the Poisson-type formula*

$$F(f) = \int\limits_{f_\tau \in \gamma_x} F(f_\tau)\, d\tau,$$

where f is univalent function of class S corresponding to the point x of the homogeneous manifold M, γ_x is a geodesic on $\Lambda_+(M) = E$, f_τ are the univalent functions from E corresponding to the points of the geodesic γ_x, τ is the natural parameter on γ_x, and F is a functional from $\Re\mathcal{O}^0(S)$.

Proof. Let f be the function of class S of the form $f(z) \equiv z$; then the univalent functions f_τ are the Koebe functions $k_\tau(z) = z/(1 - e^{i\tau}z)$ [19, 20]. Consider the analytic disk $D = \{f_a(z) = z/(1 - az)^2,\ a \in D_+\}$ lying in S. The restriction F of an analytic functional from $\Re\mathcal{O}^0(S)$ to the disk D is a harmonic function on D, and, therefore,

$$F(f_0) = \int_{f_\tau} F(f_\tau)\, d\tau$$

for all $f \in S$.

3. The infinite-dimensional geometry of the universal deformation of the complex disk

3.1. The universal deformation of the complex disk. The space A of the universal deformation of the complex disk is the set of pairs

$$(f, w), \quad f \in S, \quad w \in C \setminus ((f(D_+)^{-1} \cup \{0\}), \tag{21A}$$

and the projection onto the base has the form

$$(f, w) \mapsto f. \tag{21B}$$

The natural action of the Virasoro algebra $\mathrm{vir}_{\mathbb{C}}$ on A, introduced in [37, 38] and explicated in [36], has the form

$$\mathfrak{L}_v(f(z), a) = i \oint \Big(\frac{wf'(w)}{f(w)}\Big)^2 \frac{v(w)\, dw}{w} \Big(\frac{f^2(z)}{f(w) - f(z)}, \frac{a}{1 - af(w)}\Big) \tag{22A}$$

or

$$\begin{aligned}
L_p &= \frac{\partial}{\partial c_p} + \sum_{k \geqslant 1} (k+1) c_k \frac{\partial}{\partial c_{k+p}}, \quad p > 0\\
L_0 &= \sum_{k \geqslant 1} k c_k \frac{\partial}{\partial c_k} + w\partial_w\\
L_{-1} &= \sum_{k \geqslant 1} ((k+2)c_{k+1} - 2c_1 c_k) \frac{\partial}{\partial c_k} + 2c_1 w \frac{\partial}{\partial w} + w^2 \frac{\partial}{\partial w} + w^2 \frac{\partial}{\partial w}\\
L_{-2} &= \sum_{k \geqslant 1} ((k+3)c_{k+2} - (4c_2 - c_1^2)c_k - B_k) \frac{\partial}{\partial c_k} + (4c_2 - c_1^2) w \frac{\partial}{\partial w} + 3c_1 w^2 \frac{\partial}{\partial w} + w^3 \frac{\partial}{\partial w}\\
L_{-p} &= \frac{(-1)^p}{(p-2)!} ad^{p-2}(L_{-1}) L_{-2}, \quad p \geqslant 3.
\end{aligned}$$

This action can be exponentiated to yield an action of the Neretin semigroup. Thus the universal deformation is identified with the quotient $\mathrm{Ner} / \mathrm{Ner}^{\circ\circ}$, where $\mathrm{Ner}^{\circ\circ}$ is the subsemigroup of codimension 1 in Ner° consisting of mappings $g \in \mathcal{O}(D_-)$ with some prescribed fixed point.

The action of $\mathrm{Diff}_+(S^1)$ on the base M can be lifted to the universal deformation space A.

3.2. The universal deformation space of the complex disk as a subsymmetric space.

Definition 1. [4, 5] A pair $(\mathcal{X}, \Sigma)$, where $\mathcal{X}$ is a space and Σ the set of its involutive automorphisms (antiautomorphisms) (involutions), is called a *subsymmetric space* if a mapping

$$x \mapsto s_x$$

from X to Σ is given such that

a) $s_x(x) = x$, $s_x s_y s_x = s_{s_x y}$,

b) if for some $s \in \Sigma$ $s(x) = x$, then $s = s_x$.

Lemma. *Each subsymmetry of M can be extended to a subsymmetry of A, and this extension is* $\mathrm{Diff}_+(S^1)$*-equivariant.*

Proof. The assertion of the lemma follows from the fact that the elements of $\mathrm{Diff}_+(S^1)$ can be regarded as automorphisms of the semigroup Ner. This fact follows from the $\mathrm{Diff}_+(S^1)$-invariance of the tangent cone ner to the semigroup Ner, which lies in the Lie algebra $\mathrm{Vect}_{\mathbb{C}}(S^1)$.

Theorem 3A. *The pair $(A, \Lambda(M))$ is a subsymmetric space.*

Remark. The universal deformation space of the complex disk is projected on the symmetric space $\Lambda(M)$. Moreover, since the subsymmetry mirrors on A consist of two connected components, the universal deformation space of the complex disk is projected on the symmetric space $\Lambda_+(M)$.

In view of this fact the non-Euclidean Lagrangian $\Lambda_+(M)$ can be viewed as a "the universal deformation space of the circle."

Theorem 3B. $\Re(\mathcal{O}(A)) \simeq \mathcal{O}(\Lambda_+(M))$.

Proof. The mapping of $\Re(\mathcal{O}(A))$ into $\mathcal{O}(\Lambda_+(M))$ has the form

$$F(s,a) = \lim_{f_n \to f,\ w \to b} F(f_n, b_w),$$

where f is the function corresponding to the cut in E determined by the element $(s,a) \in \Lambda_+(M)$, and $1/b$ is the second end of the cut $f(S^1)$. The surjectivity of this mapping follows from Theorem 2 and formulas (22A).

Note that A is a symplectic [39, 40] and, moreover, a Kähler manifold. The Kähler structure can be defined by Bergman's kernel function, which is the exponential of Kähler's potential.

This kernel function is the product of the lift of Bergman's kernel function on the base M and fiberwise Bergman's kernel function. The subsymmetries of A are Kähler antiautomorphisms. In particular, the element s_- defines the subsymmetry of A of the form

$$(f(z), w) \mapsto (\overline{f(\bar{z})}, \bar{w}).$$

Mirrors of subsymmetries are Lagrangian submanifolds, and the projection of A to $\Lambda_+(M)$ defines a real $\mathrm{Diff}_+(S^1)$-invariant polarization (see [43]) on A.

The author is grateful to M. A. Semenov-Tyan-Shanskii, A. S. Schwarz, A. S. Fedenko, L. V. Sabinin, I. M. Milin, G. I. Ol′shanskii, A. M. Perelomov, A. Yu. Morozov, M. L. Kontsevich, Yu. A. Neretin, P. O. Mikheev, E. G. Emel′yanov, A. O. Radul, A. A. Roslyi, A. Yu. Alekseev, and B. A. Khesin for useful discussions of some aspects of the paper. The author thanks A. N. Rudakov and the participants of his seminar for the attention. The author is indebted to V. N. Kolokol′tsov for fruitful discussions.

References

1. Yur'ev, D. V., *Quantum conformal field theory as infinite-dimensional noncommutative geometry*, Uspekhi Mat. Nauk **46** (1991), no. 4, 115–138.
2. Yur'ev, D. V., *Infinite-dimensional geometry and quantum-field theory of strings. I. Infinite dimensional geometry of a second quantized free string*, Algebras Groups Geom. **11** (1994).
3. Yur'ev, D. V., *The vocabulary of geometry and harmonic analysis on the infinite-dimensional manifold* $\mathrm{Diff}_+(S^1)/S^1$, Adv. Soviet Math. **2** (1991), 233–247.
4. Vedernikov, V. I., Fedenko, A. S., *Symmetric spaces and their generalizations*, Algebra, Topology, Geometry, vol. 14, VINITI, Moscow, 1976, pp. 249–280.

5. Sabinin, L. V., *On geometry of the subsymmetric spaces*, Dokl. USSR Higher School, Phys.-Math. Sci. **3** (1958), 46–49.
6. Rosenfeld, B. A., *Non-Euclidean Spaces*, Nauka, Moscow, 1969.
7. Gelfand, I. M., Fuchs, D. B., *The cohomology of the Lie algebra of vector fields on a circle*, Funkts. Anal. i Prilozh. **2** (1968), no. 4, 92–93.
8. Virasoro, M. A., *Subsidiary conditions and ghosts in dual resonance models*, Phys. Rev. D. **1** (1970), 2933–2936.
9. Bott, R., *On the characteristic classes of groups of diffeomorphisms*, Enseign. Math. **23** (1977), 209–220.
10. Neretin, Yu. A., *On a complex semigroup that contains the group of diffeomorphisms of the circle*, Funkts. Anal. i Prilozh. **21** (1987), no. 2, 82–83.
11. Kontsevich, M. L., *unpublished*.
12. Segal, G., *The Definitions of the Conformal Field Theory*, Preprint MPI.
13. Neretin, Yu. A., *Holomorphic extensions of representations of the group of diffeomorphisms of the circle*, Matem. Sbornik **180** (1989), no. 5, 635–657.
14. Kirillov, A. A., *A Kähler structure on K-orbits of the group of diffeomorphisms of the circle*, Funkts. Anal. i Prilozh. **21** (1987), no. 2, 42–45.
15. Kirillov, A. A., Yur'ev, D. V., *The Kähler geometry of the infinite-dimensional homogeneous space* $M = \mathrm{Diff}_+(S^1)/\mathrm{Rot}(S^1)$, Funkts. Anal. i Prilozh. **21** (1987), no. 4, 35–46.
16. Kirillov, A. A., Yur'ev, D. V., *Representations of the Virasoro algebra by the orbit method*, J. Geom. Phys. **25** (1988), 351–363.
17. Segal, G., *Unitary representations of some infinite-dimensional groups*, Commun. Math. Phys. **80** (1981), 301–342.
18. Kirillov, A. A., *Infinite-dimensional Lie groups, their invariants and representations*, Lect. Notes Math. **970** (1982), 101–123.
19. Goluzin, G. M., *Geometric Theory of Functions of Complex Variables*, AMS, 1968.
20. Duren, P. L., *Univalent Functions*, Springer, 1983.
21. Lehto, O., *Univalent Functions and Teichmüller Spaces*, Springer, 1986.
22. Yur'ev, D. V., *On the univalency of regular functions*, Ann. Mat. Pura Appl. (4) **164** (1993), 37–50.
23. Yur'ev, D. V., *On determining univalence radius of a regular function by its Taylor coefficients*, Matem. Sbornik **183** (1992), no. 1, 45–67.
24. Bowick, M. J., Rajeev, S. G., *The holomorphic geometry of closed bosonic string theory and* Diff $(S^1)/S^1$, Nucl. Phys. B. **293** (1987), 348–384.
25. Arnold, V. I., *Mathematical Methods of Classical Mechanics*, Springer-Verlag, 1976.
26. Ginzburg, V. A., *Symplectic geometry and representations*, Funkts. Anal. i Prilozh. **17** (1983), no. 3, 75–76.
27. Guillemin, V., Sternberg, S., *Geometric Asymptotics*, AMS, 1977.
28. Weinstein, A., *Symplectic geometry*, Bull. AMS **5** (1981), 1–31.
29. Yur'ev, D. V., *Non-Euclidean geometry of mirrors and prequantization on the homogeneous Kähler manifold* $M = \mathrm{Diff}_+(S^1)/\mathrm{Rot}(S^1)$, Uspekhi Mat. Nauk **43** (1988), no. 2, 159–160.
30. Leray, J., *Analyse Lagrangienne et Mecanique Quantique*, Strasbourg, 1978.
31. Cartan, E., *Sur les domains bornes homogenes de l'espace de N variables complexes*, Oeuvres Completes **1** (1955), 1.
32. Hua Lo Ken, *Garmonicheskii Analiz Funktsii Neskol′kih Peremennyh v Klassicheskih Oblastyah (Harmonic Analysis of Functions of Several Complex Variables in Classical Domains)*, GITTL, Moscow, 1960.
33. Milin, L. M., *Odnolistnye Funktsii i Ortonormirovannye Sistemy (Univalent Functions and Orthonormal Systems)*, Nauka, Moscow, 1971.
34. Milin, L. M., *Area method in the theory of univalent functions*, Dokl. Akad. Nauk SSSR **154** (1963), no. 2, 264–267.
35. Besse, A. L., *Manifolds All Geodesics Are Closed*, Mir, Moscow, 1983.
36. Yur'ev, D. V., *A model of the Verma modules over the Virasoro algebra*, Algebra i Analiz. **2** (1990), no. 2, 209–226.
37. Manin, Yu. I., *Critical dimensions of string theories and the dualizing sheaf on the (super) curves moduli space*, Funkts. Anal. i Prilozh. **20** (1986), no. 3, 88–89.
38. Kontsevich, M. L., *The Virasoro algebra and Teichmüller spaces*, Funkts. Anal. i Prilozh. **21** (1987), no. 2, 78–79.

39. Alekseev, A., Shatashvili, S., *Path integral quantization of the coadjoint orbits of the Virasoro group and 2d gravity*, Nucl. Phys. B. **323** (1989), 719–733.
40. Alekseev, A., Shatashvili, S., *From geometric quantization to conformal field theory*, Commun. Math. Phys. **128** (1990), 197–212.
41. Krichever, I. M., *Methods of algebraic geometry in the theory of nonlinear equations*, Uspekhi Mat. Nauk **32** (1977), no. 6, 183–208.
42. Segal, G., Wilson, G., *Loop groups and equations of KdV type*, Publ. Math. IHES **61** (1985), 5–65.
43. Kirillov, A. A., *Geometric quantization*, Sovrem. Probl. Matem. Fund. Napravleniya **4** (1985), 141–178.
44. Weitsman, J., *Quantization via real polarization of the moduli space of flat connections and Chern-Simons gauge theory in genus one*, Commun. Math. Phys. **137** (1991), 175–190.
45. Weitsman, J., *Real polarization of the moduli space of flat connections on a Riemann surface*, Commun. Math. Phys. **145** (1991).
46. Jeffrey, J, C., Weitsman, J., *Bohr-Sommerfeld Orbits in the Moduli Space of Flat Connections and the Verlinde Dimension Formula*, Preprint IASSNS-HEP-91/82.
47. Jeffrey, J, C., Weitsman, J., *Half-Density Quantization of the Moduli Space of Flat Connections and Witten's Semiclassical Manifold Invariants*, Preprint IASSNS-HEP-91/94.
48. Kolokol'tzov, V. N., *Maslov index in the infinite-dimensional symplectic geometry*, Matem. Zametki **48** (1990), no. 6, 142–145.

Institute for Information Technologies, 23 Avtozavodskaya, Moscow 109280, Russia